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GEOMETRIC NONLINEARITY OF MECHANICAL BEHAVIOR AS A CONSEQUENCE OF LARGE DEFORMATIONS OF SPRINGS

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Spiral springs used in many fields of technology are usually considered linear mechanical objects, i.e., the dependence of their elongation is in proportion to the applied load. However, under large deformations, this dependence becomes nonlinear, whereas the deformations of a material from which a spring is manufactured remain small and the linearity of the properties of the material is preserved. The theory of large deformations of spiral springs that makes it possible to predict the effective elastic modulus as a function of the elongation ratio has been developed.

Keywords: *elasticity, springs, moduli, elongation ratio.*

Introduction. The idea of this investigation originates from Kirchhoff's old work [1] on large extension ratios of thin cylindrical spirals (springs). A cylindrical spiral with outside diameter D that is made of a thin spring wire of diameter d is extended by the axial thrust F. A positive compensating torque M stabilizing the spiral's angular position is applied to the wire to eliminate the unwinding (despiraling) of such a spring in the process of wire straightening.

Deformation is effected in the region of linear elasticity of a material from which a spiral is manufactured even if the spiral becomes completely straightened. The deformation of the material does not exceed 1%. This condition is observable for values of the spring index $j = D/d$ that exceed 100–150 units. It is well known that the index j of standard springs usually varies from 8 to 12 units, and their deformation does not exceed 30–40%. Consequently, the maximum extension of springs investigated in the present work is far beyond the traditional deformation range. However, such deformations are of interest to polymer physics as models of behavior of spiral macromolecules that can change from the spiral conformation to a straightened conformation. Since the extension ratios of such nonstandard springs attain several units, one should use the Hencky deformation measure [2] to evaluate the relative tensile deformation.

This work seeks to construct a model of large deformations of spiral springs that would predict the nonlinearity of their behavior. The results of such investigation can be of interest both to polymer physics and as a method of investigation of the elastic characteristics of innovation (including composite) spring materials, which are sensitive to different structural changes occurring in these materials in a prescribed thermal or mechanical treatment.

A theoretical analysis of the mechanical behavior of helical springs was first successfully made in [1]. The results of its investigations made it possible to represent the tensile force F as a function of the helix angle of the spring α, which increases with its extension. In the case where the angular position of the spring is fixed by the compensation torque M applied to it, the final result of calculating the function $F(\alpha)$ has the form

$$F = (\sin \alpha \cos^2 \alpha) \frac{K_1 + K_2}{r^2} . \tag{1}$$

The running diameter D of the spring, which diminishes with its elongation, is expressed by the value of the angle α as follows:

$$D(\alpha) = D_0 \cos \alpha . \tag{2}$$

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The radius r_0 of an undeformed spring can be expressed as $r/\cos\alpha$. Then Eq. (1) can be written as

$$F = 4 \sin\alpha \, (K_1 + K_2)/D_0^2 \,. \tag{3}$$

If the angular position of the spring is fixed by the torque M applied to it, the tensile force F is in proportion to $\sin\alpha$. The maximum force F_{max} is attained with a completely straightened spring wire when $\alpha = \pi/2$:

$$F_{max} = 4 \, (K_1 + K_2)/D_0^2 \,. \tag{4}$$

In this case we have

$$F = F_{max} \sin\alpha \,. \tag{5}$$

The angle α includes the initial value α_0 and the decrement of this angle $\Delta\alpha$:

$$\alpha = \alpha_0 + \Delta\alpha \,. \tag{6}$$

With account for (6), formula (5) can be reduced to the form

$$F = F_{max} \sin(\alpha_0 + \Delta\alpha) \,. \tag{7}$$

The running value of the spring's elongation ratio λ can be determined as follows:

$$\lambda = \sin(\alpha_0 + \Delta\alpha) \operatorname{cosec}\alpha_0 \,. \tag{8}$$

The maximum elongation ratio λ_{max} corresponding to the maximum possible extended state is determined as

$$\lambda_{max} = \operatorname{cosec}\alpha_0 \tag{9}$$

and

$$\lambda/\lambda_{max} = \sin(\alpha_0 + \Delta\alpha) \,. \tag{10}$$

The final expression for the dependence $F(\lambda)$ will take the form

$$F = F_{max} \, (\lambda/\lambda_{max}) \,. \tag{11}$$

Thus, in accordance with the results of Kirchhoff's calculations, the force F extending the spring is in the proportion to its extension ratio λ. This means that the mechanical behavior of the spring must be linear throughout its deformation range. However, this results is inconsistent with the actual situation. A more rigorous theory of extension and torsion of helical springs [3] shows that in solving exactly static equations in the region of large deformations of helical springs, the nonlinearity of their behavior appears. The geometric nonlinearity of helical springs has been investigated in [4] with variational methods. However, in the cited publication, there were no final relations that would enable one to find the tensile force as a function of the extension ratio for comparison of theoretical results to experimental data.

This determined the need for independent analysis of the problem in question from purely geometric considerations. Such analysis does not follow from the general formulation of the problem, as has been done in [3], but allows a final expression convenient for comparison to experimental data and for engineering use.

With this approach one can study multiple extension of springs whose indices exceed 50 to 100 units. Since such springs are completely restored after relieving, this is direct proof that the deformations of materials from which the springs are manufactured are limited by the linear range of deformation. This in turn points to the necessity of separating the linear mechanical behavior of the material and the nonlinear elasticity of the spring itself.

Theory. We consider the mechanical behavior of a helical spring. The polar moment of resistance (sectional modulus of torsion) W_p of spring wire of diameter d is determined as

$$W_p = 0.2d^3 . \tag{12}$$

For the spring characterized by the parameters D_0 and t_0 in the initial undeformed state, the initial angle α_0 can be determined as arcsin $[t_0/(\pi D_0)]$. The torque M required for prevention of the unwinding of the spiral will be represented as

$$M = W_p \tau , \tag{13}$$

Its value can be determined as $G\gamma$, where $\gamma = U/L$, whereas the value of the absolute shear U on the surface of the wire of length $L = \pi D_0$ within its one coil can be determined as $\varphi d/2$. In this case the shear γ can be represented as

$$\gamma = \frac{\varphi d}{2\pi D_0} = \frac{\varphi}{2\pi j} , \tag{14}$$

and the shearing stress as

$$\tau = \frac{G\varphi d}{2\pi D_0} . \tag{15}$$

As a result the torque is expressed in the form

$$M = \frac{0.032 G\varphi d^4}{D_0} . \tag{16}$$

Introducing the spring index j into this expression, we represent Eq. (16) as

$$M = \frac{0.032 G\varphi d^3}{j} . \tag{17}$$

We consider the case where the initial angle α_0 is fairly small and can be disregarded compared to $\Delta\alpha$. The coil of the spring in the maximum possible extended state represents a straight rod of length L; therefore, it angle φ of torsion will be 2π, and the helix angle α will be equal to $\pi/2$. Thus, the φ/α ratio is equal to 4. As is seen from (14), if the spring index is $j > 50$, the shear deformation will not exceed 2%, which lies within the linear region of mechanical behavior of the material from which the spring is manufactured. Between the angle of rotation φ and the decrement $\Delta\alpha$, we have the same dependence

$$\varphi = 4\Delta\alpha . \tag{18}$$

The angle $(\alpha_0 + \Delta\alpha)$ varies in the following limits:

$$\alpha_0 < (\alpha_0 + \Delta\alpha) < \pi/2 .$$

The spring diameter in the this case is in the range $D_0 > D(\alpha) > d$, and its running value is determined by analogy with (2) from the formula

$$D(\alpha) = D_0 \cos(\alpha_0 + \Delta\alpha) . \tag{19}$$

The diameter $D(\alpha)$ can be considered as the arm of application of the tensile force F. Comparing (17) and (19), we obtain an expression for the torque M and the force F:

$$M = \cos(\alpha_0 + \Delta\alpha)\, F D_0\,, \quad F = 0.127 G \Delta\alpha \sec(\alpha_0 + \Delta\alpha)\, d^2/j^2\,. \tag{20}$$

The normal stress σ in the wire cross section of area $s_d = \pi d^2/4$, which is determined as F/s_d, is equal to

$$\sigma = 0.162 S(\Delta\alpha)\, G/j^2\,, \tag{21}$$

$$S(\Delta\alpha) = \Delta\alpha \sec(\alpha_0 + \Delta\alpha)\,. \tag{22}$$

We can represent expression (21) in a more compact form as an equation that is analogous to formula (1) in structure:

$$\sigma = CS(\Delta\alpha)\,, \tag{23}$$

$$C = 0.162 G/j^2\,. \tag{24}$$

According to (20), (21), and (23), when $\Delta\alpha = 0$ we have the parameters $M = 0$, $F = 0$, and $\sigma = 0$.

The spring's effective elastic modulus E determined as σ/ε^H ($\varepsilon^H = \ln \lambda$ [1]) can be represented by the expression

$$E = S(\Delta\alpha)\, C/\ln \lambda\,. \tag{25}$$

The value of the initial elastic modulus of the spring E_0, which is determined by the dependence $\sigma(\Delta\alpha)$, has the upper bound $\Delta\alpha_{min} = 0.01$. According to (25), the quantity E_0 is calculated as follows:

$$E_0 = S(\Delta\alpha_{min})\, C/\ln \lambda_{min}\,, \tag{26}$$

$$S(\Delta\alpha_{min}) = 0.01 \sec(\alpha_0 + 0.01)\,. \tag{27}$$

The minimum elongation ratio λ_{min}, according to (8), is determined as

$$\lambda_{min} = \sin(\alpha_0 + 0.01)\, \mathrm{cosec}(\alpha_0)\,. \tag{28}$$

Expression (26) can be represented in a more compact form

$$E_0 = C q_{min}\,, \tag{29}$$

where q_{min} is determined only by the initial angle α_0:

$$q_{min} = S(\Delta\alpha_{min})/\ln \lambda_{min}\,. \tag{30}$$

When $\alpha_0 < 0.5$ the spring constant q_{min} is nearly equal to $\tan \alpha_0$.

The running value of the spring's elastic modulus E normalized to its initial value E_0 can be represented by the expression

$$E/E_0 = S(\Delta\alpha)\, (q_{min} \ln \lambda)^{-1}\,. \tag{31}$$

If the E_0 and α_0 values are known, we can calculate, from (29) and (30), the constant C and then determine the elastic modulus of the material G from (24):

$$G = C j^2/0.162\,. \tag{32}$$

Substituting (18) into (14), we obtain an expression for determining the maximum shear:

TABLE 1. Parameters of Steel Springs with Different Values of the Indices

Paremeters	Spring numbers						
	1	2	3	4	5	6	7
D, mm	2	1.2	0.2	1	0.5	0.4	2.9
j	150	147	160	91	123	159	124
E_0, kPa	234	828	541	2000	1755	1413	2530
α_0, rad	0.14	0.3	0.32	0.363	0.387	0.48	0.712
G, GPa	277	319	230	249	380	361	216
$\frac{G}{E}$	0.9	1.28	0.92	1.0	1.52	1.44	0.86
γ_{max}, %	0.61	0.55	0.5	0.84	0.61	0.44	0.44

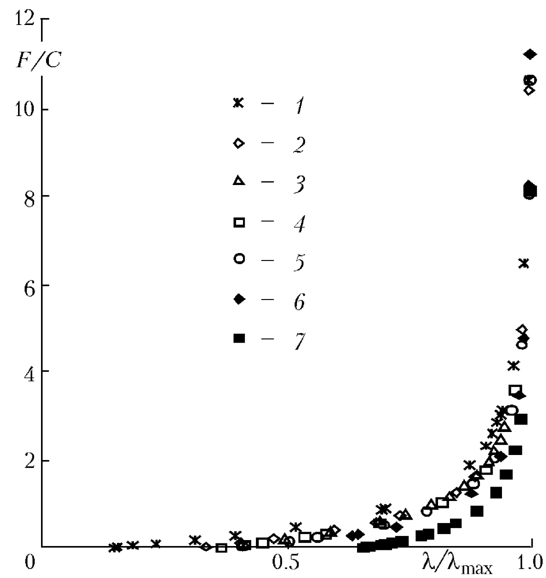

Fig. 1. Theoretical values of the tensile force F determined from formula (20) and normalized to the spring constant C (calculated from formula (24)) vs. relative elongation ratio λ/λ_{max} for the seven investigated springs. The notation of the dots corresponds to the spring numbers indicated in Table 1.

$$\gamma_{max} = 2\Delta\alpha_{max}/(\pi j)\ . \qquad (33)$$

If, on extending the spring, its wire is completely straightened, we can determine $\Delta\alpha_{max}$ as $(\pi/2 - \alpha_0)$. In this case expression (33) is reduced to the form

$$\gamma_{max} = (\pi - 2\alpha_0)/(\pi j)\ . \qquad (34)$$

If the quantity α_0 can be disregarded, we have $\gamma_{max} = 1/j$. Thus, when $j > 100$ the value of γ_{max} cannot exceed 1%.

To evaluate the influence of the parameters of different springs on the behavior of the stress-deformation curves we consider the values of the stresses σ, which have been normalized to the constant C (24), plotted as functions of the experimental values (calculated from formula (8)) of the elongation ratios λ, which have been normalized to the maximum value λ_{max}. We will consider the actual springs whose parameters are given in Table 1.

Calculation results are given in Fig. 1, from which it is clear that the $\sigma/C = f(\lambda/\lambda_{max})$ plots become shorter with increase in the initial angle α_0, and the intensity of their growth is sharply enhanced with deformation. The plots approach the abscissa axis with decrease in the deformation, and the vertical axis corresponding to $\lambda/\lambda_{max} = 1$ with increase in the deformation.

Conclusions. The circumstance that large deformations can be responsible for the nonlinear behavior of a material is well known in the theory of high elasticity of rubbers. However, in this work, we have investigated another case, namely, that of preservation of the linearity of the material's behavior and of occurrence of nonlinear phenomena in the entire structure. It has been shown that the nonlinearity of the mechanical behavior of spiral springs in the range of large elongation ratios can be explained by the fact itself of large deformations, with the deformations of the spring material being small and limited by the linear range. This result points to the necessity of differentiating between the linearity of behavior of a material and the nonlinearity of behavior of a structure manufactured from this material. The obtained theoretical results have demonstrated the role of different factors in the nonlinearity of behavior of spiral springs with large indices (ratio of the diameter of the spiral to the diameter of wire from which it is manufactured). The obtained results can be of importance in analyzing the deformation of spiral macromolecules in polymer physics and analyzing the mechanical behavior of spiral springs manufactured from new materials.

NOTATION

C, normalizing constant, Pa; D, outside diameter of the cylindrical spiral, m; d, diameter of the spring wire, m; D_0, diameter of the axial line of an undeformed spring, m; E, effective elastic modulus of the spring, Pa; E_0, initial elastic modulus of an undeformed spring, Pa; F, axial thrust extending the spring, N; F_{max}, maximum tensile force in the completely straightened spring wire, N; G, elastic modulus of the spring material, Pa; j, spring index; K_1 and K_2, spring constants, N·m^2; L, wire length within one coil, m; M, positive compensating torque, N·m; q_{min}, spring constant; r, spiral radius, m; r_0, radius of an undeformed spiral, m; s_d, area of the wire cross section, m^2; $S(\Delta\alpha)$, nonlinear function of the angle decrement; t_0, pitch of an undeformed spiral, m; U, absolute shear on the wire surface, m; W_p, polar moment of resistance (sectional modulus of torsion) of the spring wire, m^3; α, helix angle of the spring, rad; α_0, helix angle of an undeformed spring, rad; $\Delta\alpha$, angle decrement, rad; $\Delta\alpha_{min}$, upper boundary of the initial portion, rad; $\Delta\alpha_{max}$, maximum value of the angle decrement, rad; γ, relative shear; γ_{max}, maximum shear; ε^H, Hencky deformation measure; λ, elongation ratio of the spring; λ_{max}, maximum elongation ratio; λ_{min}, elongation ratio at the boundary of the initial portion; σ, normal stress, Pa; τ, shearing stress in the surface wire layer, Pa; φ, angle of torsion within a coil. Subscripts: 0, initial; p, polar; 1 and 2, ordinal numbers.

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